Exercise 3D

1 a $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + \mathrm{i}\sin^3\theta$

de Moivre's Theorem.

$= \cos^3\theta + {}^3\mathrm{C}_1\cos^2\theta(\mathrm{i}\sin\theta) + {}^3\mathrm{C}_2\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$

Binomial expansion.

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta + 3\mathrm{i}^2\cos\theta\sin^2\theta + \mathrm{i}^3\sin^3\theta$

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

Hence,

$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

Equating the imaginary parts gives,

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (as required)

b $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$

de Moivre's Theorem.

$= \cos^5\theta + {}^5\mathrm{C}_1\cos^4\theta(\mathrm{i}\sin\theta) + {}^5\mathrm{C}_2\cos^3\theta(\mathrm{i}\sin\theta)^2 + {}^5\mathrm{C}_3\cos^2\theta(\mathrm{i}\sin\theta)^3 + {}^5\mathrm{C}_4\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$

Binomial expansion.

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta + 10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$

Hence,

$\cos 5\theta + \mathrm{i}\sin 5\theta = \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$

Equating the imaginary parts gives,

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(\cos^2\theta)^2\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Hence, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (as required)

1 c $(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$

$$= \cos^7\theta + {}^7C_1\cos^6\theta(i\sin\theta) + {}^7C_2\cos^5\theta(i\sin\theta)^2 + {}^7C_3\cos^4\theta(i\sin\theta)^3 + {}^7C_4\cos^3\theta(i\sin\theta)^4 + {}^7C_5\cos^2\theta(i\sin\theta)^5 + {}^7C_6\cos\theta(i\sin\theta)^6 + (i\sin\theta)^7$$

de Moivre's Theorem.

Binomial expansion.

$$= \cos^7\theta + 7i\cos^6\theta\sin\theta + 21i^2\cos^5\theta\sin^2\theta + 35i^3\cos^4\theta\sin^3\theta + 35i^4\cos^3\theta\sin^4\theta + 21i^5\cos^2\theta\sin^5\theta + 7i^6\cos\theta\sin^6\theta + i^7\sin^7\theta$$

Hence,

$$\cos 7\theta + i\sin 7\theta = \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i^3\cos^4\theta\sin^3\theta + 35i^4\cos^3\theta\sin^4\theta + 21i^5\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

Equating the imaginary parts gives,

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta) - 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Hence, $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos^5\theta$ (as required)

1 d Let $z = \cos\theta + \text{i}\sin\theta$

$$\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$$

$z + \dfrac{1}{z} = 2\cos\theta$

$$= z^4 + {}^4\text{C}_1\, z^3\left(\frac{1}{z}\right) + {}^4\text{C}_2\, z^2\left(\frac{1}{z}\right)^2 + {}^4\text{C}_3\, z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z^2\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$z^n + \dfrac{1}{z^n} = 2\cos n\theta$

So, $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$16\cos^4\theta = 2(\cos 4\theta + 4\cos 2\theta + 3)$

$\cos^4\theta = \dfrac{2}{16}(\cos 4\theta + 4\cos 2\theta + 3)$

Therefore, $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (as required)

e Let $z = \cos\theta + \text{i}\sin\theta$

$z - \dfrac{1}{z} = 2\text{i}\sin\theta$

$$\left(z-\frac{1}{z}\right)^5 = (2\text{i}\sin\theta)^5 = 32\text{i}^5\sin^5\theta = 32\text{i}\sin^5\theta$$

$$= z^5 + {}^5\text{C}_1\, z^4\left(-\frac{1}{z}\right) + {}^5\text{C}_2\, z^3\left(-\frac{1}{z}\right)^2 + {}^5\text{C}_3\, z^2\left(-\frac{1}{z}\right)^3 + {}^5\text{C}_4\, z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5}$$

$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$= 2\text{i}\sin 5\theta - 5(2\text{i}\sin 3\theta) + 10(2\text{i}\sin\theta)$$

$z^n + \dfrac{1}{z^n} = 2\text{i}\sin n\theta$

So, $32\text{i}\sin^5\theta = 2\text{i}\sin 5\theta - 10\text{i}\sin 3\theta + 20\text{i}\sin\theta$ $(\div 2\text{i})$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Therefore, $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

2 a Let $z=\cos\theta+\mathrm{i}\sin\theta$ then we have $z^5+z^{-5}=2\cos 5\theta$ hence

$$\cos 5\theta=\frac{1}{2}\left(z^5+z^{-5}\right)$$

We have

$$z^5=\left(\cos\theta+\mathrm{i}\sin\theta\right)^5$$
$$=\left(\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta-10\mathrm{i}\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+\mathrm{i}\sin^5\theta\right)$$

and

$$z^{-5}=\left(\cos\theta-\mathrm{i}\sin\theta\right)^5$$
$$=\left(\cos^5\theta-5\mathrm{i}\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta+10\mathrm{i}\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta-\mathrm{i}\sin^5\theta\right)$$

Hence

$$\cos 5\theta=\tfrac{1}{2}\left(2\cos^5\theta-20\cos^3\theta\sin^2\theta+10\cos\theta\sin^4\theta\right)$$
$$=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$$
$$=\cos^5\theta-10\cos^3\theta\left(1-\cos^2\theta\right)+5\cos\theta(1-\cos^2\theta)^2$$
$$=\cos^5\theta-10\cos^3\theta+10\cos^5\theta+5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$
$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

b We have that $\cos 3\theta=4\cos^3\theta-3\cos\theta$ and we wish to solve

$$\cos 5\theta+5\cos 3\theta=0$$

Using the above identities the equation becomes

$$16\cos^5\theta-20\cos^3\theta+5\cos\theta+5\left(4\cos^3\theta-3\cos\theta\right)=0$$

Which simplifies to

$$16\cos^5\theta-10\cos\theta=0$$

Hence we either have $\cos\theta=0$ in which case $\theta=\frac{\pi}{2}=1.571$ or we have

$$16\cos^4\theta=10$$

Hence

$$\cos\theta=\pm\sqrt[4]{\frac{5}{8}}$$

Which implies we have $\theta=0.475$ or $\theta=2.666$

3 Let $z = \cos\theta + \mathrm{i}\sin\theta$

a $\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

$z - \frac{1}{z} = 2\cos\theta$

$= z^6 + {}^6C_1 z^5\left(\frac{1}{z}\right) + {}^6C_2 z^4\left(\frac{1}{z}\right)^2 + {}^6C_3 z^3\left(\frac{1}{z}\right)^3 + {}^6C_4 z^2\left(\frac{1}{z}\right)^4 + {}^6C_5\, z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$

$= z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$

$= z^6 - 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\sin 2\theta) + 20$

$z^n + \frac{1}{z^n} = 2\cos\theta$

So, $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ (as required)

b $\int_0^{\frac{\pi}{6}} \cos^6\theta\,\mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{6}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$

$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} + \frac{6\sin 4\theta}{4} + \frac{15\sin 2\theta}{2} + 10\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{32}\left[\left(\frac{\sin(\pi)}{6} + \frac{6\sin\left(\frac{2\pi}{3}\right)}{4} + \frac{15\sin\left(\frac{\pi}{3}\right)}{2} + \frac{10\pi}{6}\right) - (0)\right]$

$= \frac{1}{32}\left[0 + \frac{3}{2}\frac{\sqrt{3}}{2} + \frac{15}{2}\frac{\sqrt{3}}{2} + \frac{5\pi}{3}\right]$

$= \frac{1}{32}\left[\frac{3}{4}\sqrt{3} + \frac{15}{4}\sqrt{3} + \frac{5\pi}{3}\right]$

$= \frac{1}{32}\left[\frac{9}{2}\sqrt{3} + \frac{5\pi}{3}\right]$

$= \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$

$\therefore \int_0^{\frac{\pi}{6}} \cos^6\theta\,\mathrm{d}\theta = \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$

$a = \frac{5}{96},\ b = \frac{9}{64}$

4 a We wish to show that

$32\cos^2\theta\sin^4\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

Let us start with the right hand side of the equation, letting $z = \cos\theta + i\sin\theta$ we have

$$\cos 6\theta = \tfrac{1}{2}\left(z^6 + z^{-6}\right)$$
$$= \tfrac{1}{2}\left(2\cos^6\theta - 30\cos^4\theta\sin^2\theta + 30\cos^2\theta\sin^4\theta - 2\sin^6\theta\right)$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$\cos 4\theta = \tfrac{1}{2}\left(z^4 + z^{-4}\right)$$
$$= \tfrac{1}{2}\left(2\cos^4\theta - 12\cos^2\theta\sin^2\theta + 2\sin^4\theta\right)$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Hence the right hand side becomes

$$\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta - 2\left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right)$$
$$-\left(\cos^2\theta - \sin^2\theta\right) + 2$$
$$= \cos^2\theta\left(1-\sin^2\theta\right)^2 - 15\left(1-\sin^2\theta\right)\cos^2\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \left(1-\cos^2\theta\right)\sin^4\theta$$
$$-2\cos^4\theta + 12\cos^2\theta\sin^2\theta - 2\sin^4\theta - \cos^2\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta + \cos^2\theta - 2\cos^2\theta\sin^2\theta - 15\cos^2\theta\sin^2\theta - \sin^4\theta - 2\cos^4\theta + 12\cos^2\theta\sin^2\theta$$
$$-2\sin^4\theta - \cos^2\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - 5\cos^2\theta\sin^2\theta - 3\sin^4\theta - 2\cos^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - \cos^2\theta\left(5\sin^2\theta + 2\cos^2\theta\right) - 3\sin^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - \left(1-\sin^2\theta\right)\left(3\sin^2\theta + 2\right) - 3\sin^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta$$

b We have

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta\,d\theta = \frac{1}{32}\int_0^{\frac{\pi}{3}} (\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)\,d\theta$$
$$= \frac{1}{32}\left[\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta - \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{32}\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{2\pi}{3}\right) = \frac{\pi}{48}$$

5 a We wish to compute $\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta$, let $z = \cos\theta + i\sin\theta$ then we have

$\sin\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)$ so that

$$\sin^6\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^6 = \frac{-1}{64}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$$

$$= \frac{-1}{64}\left(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\right)$$

$$= \frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

Hence the integral becomes

$$\frac{1}{32}\int_0^{\frac{\pi}{2}}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)d\theta = \frac{1}{32}\left[\frac{-1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta - \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5\pi}{32}$$

b We wish to compute $\int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta \, d\theta$ we have

$$\sin^2\theta\cos^4\theta = \sin^2\theta\left(1 - \sin^2\theta\right)^2 = \sin^6\theta - 2\sin^4\theta + \sin^2\theta$$

From the previous part we know that

$$\sin^6\theta = \frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

and

$$\sin^4\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^4 = \frac{1}{16}\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$$

$$= \frac{1}{16}\left(2\cos 4\theta - 8\cos 2\theta + 6\right) = \frac{1}{32}\left(4\cos 4\theta - 16\cos 2\theta + 12\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right) = \frac{1}{32}\left(16 - 16\cos 2\theta\right)$$

Hence we have

$$\sin^2\theta\cos^4\theta = \sin^6\theta - 2\sin^4\theta + \sin^2\theta$$

$$= \frac{1}{32}\left(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2\right)$$

So the integral becomes

$$\frac{1}{32}\int_0^{\frac{\pi}{4}}\left(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2\right)d\theta = \frac{1}{32}\left[-\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{32}\left(\frac{1}{6} + \frac{1}{2} + \frac{\pi}{2}\right) = \frac{1}{64}\left(\frac{4}{3} + \pi\right) = \frac{4 + 3\pi}{192} = \frac{\pi}{64} + \frac{1}{48}$$

5 c We wish to compute

$$\int_0^{\frac{\pi}{6}} \sin^3\theta\cos^5\theta\,d\theta$$

Let $z = \cos\theta + i\sin\theta$ then we have

$$\sin^3\theta\cos^5\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^3\left(\frac{1}{2}\left(z + z^{-1}\right)\right)^5$$

$$= \frac{i}{256}\left(z^3 - 3z + 3z^{-1} - z^{-3}\right)\left(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}\right)$$

$$= \frac{i}{256}\left(z^8 + 2z^6 - 2z^4 - 6z^2 + 6z^{-2} + 2z^{-4} - 2z^{-6} - z^{-8}\right)$$

$$= \frac{i}{256}\left(2i\left(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right)\right)$$

$$= \frac{-1}{128}\left(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right)$$

Hence the integral becomes

$$\frac{-1}{128}\int_0^{\frac{\pi}{6}}\left(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right)d\theta$$

$$= \frac{-1}{128}\left[\frac{-1}{8}\cos 8\theta - \frac{1}{3}\cos 6\theta + \frac{1}{2}\cos 4\theta + 3\cos 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{-1}{128}\left(\left(\frac{1}{16} + \frac{1}{3} - \frac{1}{4} + \frac{3}{2}\right) - \left(\frac{-1}{8} - \frac{1}{3} + \frac{1}{2} + 3\right)\right) = \frac{-1}{128}\left(\frac{79}{48} - \frac{146}{48}\right) = \frac{67}{6144}$$

6 a Let $z = \cos\theta + i\sin\theta$ then we have

$$\cos 6\theta = \tfrac{1}{2}\left(z^6 + z^{-6}\right) = \tfrac{1}{2}\left(\left(\cos\theta + i\sin\theta\right)^6 + \left(\cos\theta - i\sin\theta\right)^6\right)$$

Noting that odd powers will cancel this simplifies to

$$= \tfrac{1}{2}\left(2\cos^6\theta - 30\cos^4\theta\sin^2\theta + 30\cos^2\theta\sin^4\theta - 2\sin^6\theta\right)$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta\left(1 - \cos^2\theta\right) + 15\cos^2\theta\left(1 - \cos^2\theta\right)^2 - \left(1 - \cos^2\theta\right)^3$$

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

6 b We wish to solve

$32x^6 - 48x^4 + 18x - \frac{3}{2} = 0$

We use the substitution $x = \cos\theta$ so that the equation becomes

$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - \frac{3}{2} = 0$

i.e.

$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \frac{1}{2}$

Hence

$\cos 6\theta = \frac{1}{2}$

The general solution to this is given by

$6\theta = \pm\frac{\pi}{3} + 2k\pi$ where k is any integer i.e.

$\theta = \pm\frac{\pi}{18} + \frac{k\pi}{3}$

Trying both choices of sign and varying k gives the following values of $x = \cos\theta$

$x = \pm 0.985$

$x = \pm 0.342$

$x = \pm 0.643$

But these are 6 distinct solutions and since the polynomial has order 6 there are at most 6 unique solutions hence these are all solutions.

7 a $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ — de Moivre's Theorem.

$$= \cos^4\theta + {}^4C_1\cos^3\theta(i\sin\theta) + {}^4C_2\cos^2\theta(i\sin\theta)^2 + {}^4C_3\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

Binomial expansion.

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Hence,

$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$ **(1)**

Equating the imaginary parts of **(1)** gives:

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (as required)

7 b Equating the real parts of **(1)** gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \qquad \frac{(\sin 4\theta \div \cos^4\theta)}{(\cos 4\theta \div \cos^4\theta)}$$

$$= \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{\frac{4\cancel{\cos^3\theta}}{\cancel{\cos^3\theta}}\frac{\sin\theta}{\cos\theta} - \frac{4\cancel{\cos\theta}\sin^3\theta}{\cancel{\cos\theta}\cos^3\theta}}{\frac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \frac{6\cancel{\cos^2}\theta\sin^2\theta}{\cancel{\cos^2}\theta\cos^2\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Therefore, $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (as required)

c $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

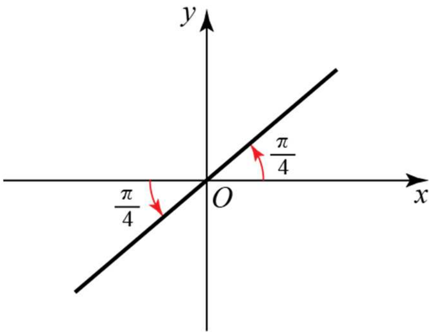
$1 = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$ **(2)**

Let $x = \tan\theta$; then

(2) $\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$

$\tan 4\theta = 1$ ← From part **b**.

$\alpha = \dfrac{\pi}{4}$



$$4\theta = \left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots\right\}$$

$$\theta = \left\{\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \ldots\right\}$$

$\therefore x = \tan\theta = \tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$

$x = 0.19891\ldots, 1.49660\ldots, -5.02733\ldots, -0.66817\ldots,$

$x = 0.20, 1,50, -5.03, -0.67$ (2 d.p.)